FOURIER ANALYSIS OF DISCRETE-TIME SIGNALS

Now that we have examined the Fourier series and Fourier transforms of continuous signals we want to develop the Fourier series and transforms for discrete signals. We know that Fourier analysis allows us to expand a time domain signal into a different representation which indicates the frequency or spectral content of the signal. This frequency domain representation of the signal provides convenience in determining the time domain output of a system to an arbitrary input, and at the same time gives insight into the frequency domain representations of the input, the system response, and the output of the system. We now extend these Fourier analysis concepts to discrete time signals. As a first step let's define some terms that sometimes get confusing.

Before we can begin analysis of a discrete time signal we need to understand what this signal is and where it comes from. A discrete time signal is simply a continuous time signal which has been sampled. If a function, x(t), is continuous at t = nT, where T is the sampling interval, then a sample of x(t) at time nT is expressed as

$$x(n) = x(t)\delta(t-nT) = x(nT)\delta(t-nT), \tag{1}$$

where the delta function is used to hold the proper place on the time axis. This multiplication yields an amplitude equal to the value of x(t) at t = nT. If we now let n be a sequence of integer values, rather than one specific sample time, i.e., n = 0, ± 1, ± 2, ... then

$$x(n) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT). \tag{2}$$

We will use this discrete time signal x(n) and look at three distinctly different transforms: the discrete time Fourier series, which is a series representation for periodic signals just as before; the discrete time Fourier transform (DTFT) which is continuous function which represents the frequency transform of the sampled signal; and the discrete Fourier transform (DFT) which is a discrete signal used to represent

the frequency transform of the sampled signal. The DFT is of the form readily used by computers in digital signal processing and has become extremely valuable in that area.

A. FOURIER SERIES FOR PERIODIC DISCRETE TIME SIGNALS

Recall that for periodic continuous time signals we developed the Fourier series to represent them. We will now develop the Fourier series for a periodic discrete time signal. A signal, x(n), is periodic if for some integer N

$$x(n) = x(n+N). \tag{3}$$

A consequence of this relationship that we will make great use of is that once we know these N values of x we know all values of x for all values of n since x repeats itself after N samples.

For a continuous time cosine signal we know that

$$x(t) = a \cos\left(\frac{2\pi t}{T}\right), \tag{4}$$

where T is the period of the signal and (small omega) $\omega_0 = 2\pi/T$. For the sampled signal letting t=nT (=n), and the period N we have

$$x(n) = a_n \cos\left(\frac{2\pi n}{N}\right). \tag{5}$$

We can define a fundamental frequency for the discrete time signal, calling it by large omega

$$\Omega_0 = \frac{2\pi}{N}. \tag{6}$$

Notice that since we dropped T from nT (T has units of seconds), that the units of Ω are radians rather than radians per second as with ω.

Using the exponential form of the Fourier series recall that we found that

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_0 t}, \qquad \textbf{(7)}$$

where

$$c_n = \frac{1}{T} \int_{-\infty}^{\infty} x(t) \, e^{-jn\omega_0 t} \, dt. \qquad \textbf{(8)}$$

By analogy let t → n, ω → Ω, (and letting k become the summation indicator) Eq. 7 becomes

$$x(n) = \sum_{k=0}^{N-1} a_k \, e^{jk\Omega_0 n}. \qquad \textbf{(9)}$$

Notice that the summation extends from zero to N-1 (yielding N values for x(n)) since x(n) repeats after N, the period of x, and that n extends over all values (representing time).

Carrying the analogy to completion, we let T → N and Eq. 8 becomes

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \, e^{-jk\Omega_0 n}. \qquad \textbf{(10)}$$

Notice that the infinite integral becomes a summation over N values because the values repeat after those N values. This completes the analogy so that we have Equations 9 and 10 to define the discrete time Fourier series.

We can see that the Fourier series is a finite sum (Eq. 9 sums only N different values) rather that the infinite sum for the continuous case. Also notice that Eq. 9 gives us the frequency components of x(n), e.g., for k = 0, we have

$$x(n)\big|_{k=0} = a_0 \, e^{0} = a_0, \qquad \textbf{(11)}$$

which is the dc value of x(n). Similarly, for k = 1

$$x(n)\big|_{k=1} = a_1 \; e^{j\Omega_0 n}, \tag{12}$$

which is the value for the fundamental frequency, Ω_0. The values of the remaining harmonics can be found in the same manner. Again unlike the continuous case, there are not an infinite number of harmonics; k represents the harmonic number and extends from 0 to N-1, so that there are N total harmonics including dc and the fundamental.

We asserted that through analogy with the continuous case that the equations of Eq. 9 and 10 represent the Fourier series for the discrete case. To prove if this is true let's substitute Eq. 10 directly into Eq. 9 to see what we get for x(n),

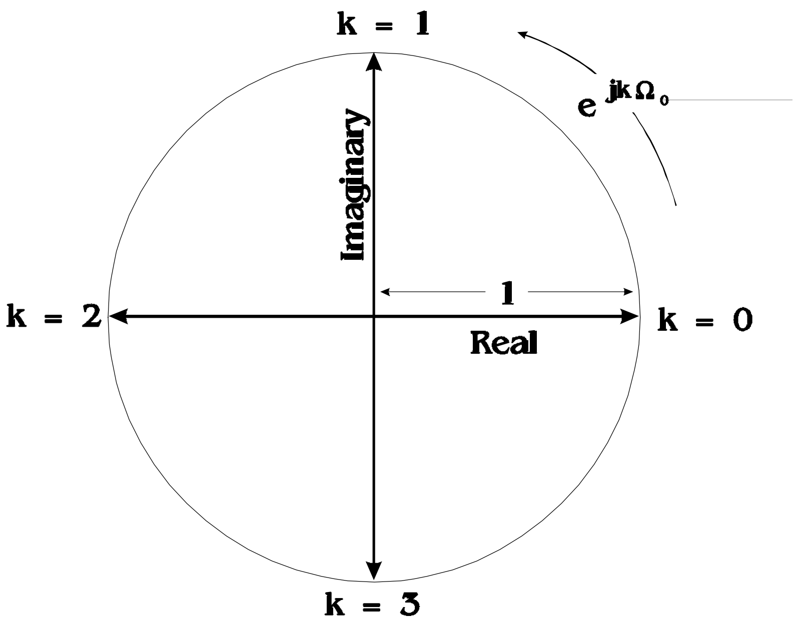
$$\begin{aligned} x(n) &= \sum_{k=0}^{N-1} a_k \; e^{jk\Omega_0 n} \\ &\stackrel{?}{=} \sum_{k=0}^{N-1} \left(\frac{1}{N} \sum_{m=0}^{N-1} x(m) \; e^{-jk\Omega_0 m} \right) \; e^{jk\Omega_0 n} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} x(m) \sum_{k=0}^{N-1} \; e^{jk\Omega_0 (n-m)}. \end{aligned} \tag{13}$$

To evaluate the last summation of Eq. 13, recall that the integral of the conjugate multiplication of orthogonal functions over one period is zero. The same holds true over one period of summation of discrete orthogonal functions.

Let's see why this is true. As an example, let N = 4. With N = 4, $\Omega_0 = 2\pi/4 = \pi/2$. The summation of a complex exponential, $e^{jk\Omega_0}$, over one period will be

$$\sum_{k=0}^{3} e^{jk\frac{\pi}{2}} = e^{j0} + e^{j\frac{\pi}{2}} + e^{j\pi} + e^{j\frac{3\pi}{2}} = 0. \tag{14}$$

In graphical form you can see these four vectors in the real-imaginary plane as shown below. Notice that since the complex exponential has a magnitude of unity, the vectors in the real-imaginary plane have unit magnitude. The vector at k = 0 is real and has a value of 1. The vector at k = 2 is real and has a value of -1. The two real values therefore cancel each other. Similarly, the two imaginary values cancel each



other so that the net of the summation is zero as predicted in Eq. 14 (and as we expected since we know that the summation over one period of a periodic signal is zero.

Now let's raise the exponential by a power of n. With N = 4 we now have

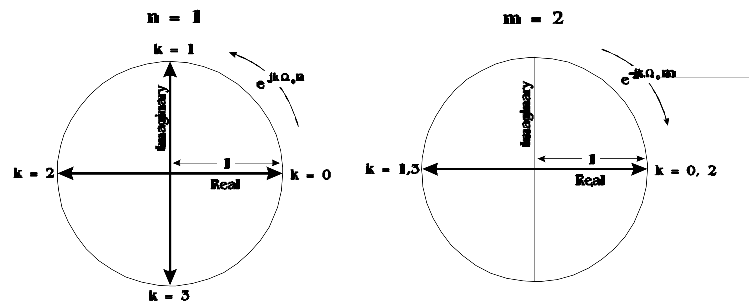
$$\left(e^{jk\Omega_0}\right)^n = e^{jk\frac{\pi}{2}n}. \quad \textbf{(15)}$$

We want to multiply this exponential by the complex conjugate of the orthogonal vector in m and sum over one period, or

$$\sum_{k=0}^{3} e^{jk\frac{\pi}{2}n}\left(e^{jk\frac{\pi}{2}m}\right)^* = \sum_{k=0}^{3} e^{jk\frac{\pi}{2}n}\ e^{-jk\frac{\pi}{2}m}. \quad \textbf{(16)}$$

In graphical form we see these two vectors below for the case when n = 1 and m = 2. For the n vector we sum

k	**0**	**1**	**2**	**3**
$e^{jk\Omega_0 n}$	**1**	$e^{j\pi/2}$	**-1**	$e^{j3\pi/2}$ **= 0**



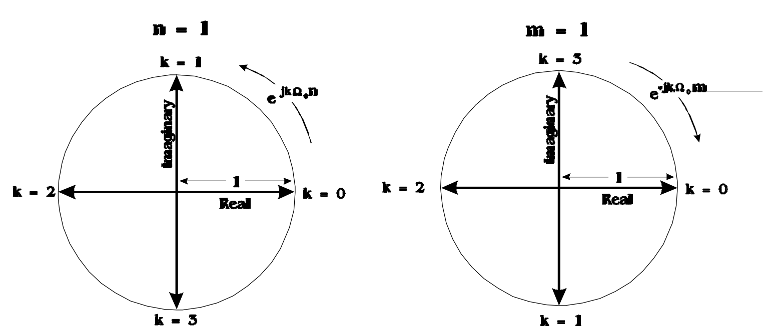
just as before. For the m vector we sum 1, -1, 1, -1. To get the sum of the multiplication of the two vectors we add

k	**0**		**1**		**2**		**3**
	$(1)(1)$	$+$	$(-1)(e^{j\pi/2})$	$+$	$(1)(-1)$	$+$	$(-1)(e^{j3\pi/2}) = 0.$

This tells us that if $m \neq n$ the sum over one period of the exponentials is zero. This is what was predicted due to orthogonality. But also by orthogonality we should have a nonzero value if the exponentials are equal, i.e., $m = n$. This can be seen in the figure below. Summing the multiplication of the two vectors yields

k	**0**	**1**	**2**	**3**
	$(1)(1)$	$e^{j\pi/2}e^{j3\pi/2}$	$(-1)(-1)$	$e^{j3\pi/2}e^{j\pi/2} = 4.$

Therefore, the sum over one period of two exponentials, one raised to n and the



other to -m when m = n, is N. For shorthand notation we use w_N to represent $e^{jk_\circ}$. Using this notation we see that

$$\sum_{k=0}^{N-1} \left(w_N^{k}\right)^n \left(w_N^{-k}\right)^m = \begin{cases} 0, & m \neq n \\ N, & m = n \end{cases} \tag{17}$$

B. DISCRETE TIME FOURIER TRANSFORM

Because we know that not all signals are periodic, we must find a method of obtaining the frequency transform of discrete nonperiodic signals. In continuous signal analysis we let the period approach infinity and determined the frequency domain description of the continuous signal, which was the Fourier transform. The transformation led to the transform of the time domain signal and the inverse transform which allowed us to return to the time domain. These two equations were determined to be

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega, \tag{18}$$

and

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt. \tag{19}$$

Using the direct analogy methods that we used for the discrete Fourier series, we can substitute for Eq. 18 as

$$x(n) = \frac{1}{2\pi} \int_{0}^{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega. \tag{20}$$

Making the direct substitutions of n for t and Ω for ω we see that Eq. 20 is identical to Eq. 18 except for the limits of integration. The discrete version of Eq. 20 need only be integrated from 0 to 2π since the frequency response of the discrete time signal is periodic at 2π. This arises from the fact that the complex exponentials are periodic at 2π.

Next substituting the discrete variables into Eq. 19 we get

$$X(\Omega) \;=\; \sum_{n=-\infty}^{\infty} x(n)\; e^{-j\Omega n}. \tag{21}$$

We see that Eq. 21 is identical to Eq. 19 except that the integral over t became a summation over n. This follows from the fact that x(n) is zero except when t = nT, so that the infinite integral becomes the infinite summation.

Eq. 21 is the discrete time Fourier transform (DTFT) and Eq. 20 is the inverse discrete time Fourier transform (IDTFT). The only thing left is to verify the validity of these substitutions.

To see if these substitutions are valid, lets substitute Eq. 21 directly into Eq. 20 and see what we get. Changing the index of summation from n to m this will give

$$x(n) \;\stackrel{?}{=}\; \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{m=-\infty}^{\infty} x(m)\; e^{-j\Omega m} \right] e^{j\Omega n}\; d\Omega. \tag{22}$$

Rearranging we have Eq. 22 into the form of

$$x(n) \;\stackrel{?}{=}\; \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} x(m) \int_0^{2\pi} e^{-j\Omega m}\; e^{j\Omega n}\; d\Omega. \tag{23}$$

Knowing the rules of orthogonal function integrations over one period, we see that the integral is zero except when m = n, when the integral will be 2π. Equation 23 reduces to

$$x(n) \;\stackrel{?}{=}\; \left[\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} x(m)\; 2\pi \right]_{m=n} \tag{24}$$

and the summation will therefore be zero except when m = n, the 2π's will cancel and the right side does indeed equal x(n).

Recall from continuous-time analysis that the Fourier transform of a shifted delta function is

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t-t_0)\; e^{-j\omega t}\; dt = e^{-j\omega t_0}. \tag{25}$$

If we extend this to a string of delta functions (describing perhaps a discrete square wave) from t = -N_1 to N_1, we would get

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} \sum_{k=-N_1}^{N_1} \delta(t-kT)\; e^{-j\omega kT}\; dt \\ &= \sum_{k=-N_1}^{N_1} \int_{-\infty}^{\infty} \delta(t-kT)\; e^{-j\omega kT}\; dt = \sum_{k=-N_1}^{N_1} e^{-j\omega kT}. \end{aligned} \tag{26}$$

Now, if we substitute our discrete time equivalents, i.e., $\omega \rightarrow \Omega$, k → n, and nT → n, we will have

$$X(\Omega) = \sum_{n=-N_1}^{N_1} e^{-j\Omega n}. \tag{27}$$

You can see that this is exactly what we will get by direct evaluation using Eq. 21.

The fact that the DTFT is periodic will become more evident when we examine the sampling theorem.

We have a condition which ensures convergence, just as we had the Dirichlet conditions in the continuous domain. The DTFT for a signal x(n) exists if the sum of the Eq. 21 converges for all real values of Ω. Therefore, the series will converge if

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty. \tag{28}$$

Note, that like the Dirichlet conditions this is not a necessary condition, just that if this condition is satisfied the existence of the transform is guaranteed. A simple example of this is a constant (i.e., an infinite string of delta functions) will produce a transform similar to Eq. 27 with the limits of the summation equal to ± ∞, so that

$$x(n) = 1 \rightleftharpoons \sum_{n=-\infty}^{\infty} e^{-j\Omega n}. \tag{29}$$

We saw for the discrete time Fourier series that we obtained discrete spectral lines--only discrete frequencies, which are represented by delta function in the frequency domain. Therefore, we now want to know what a string of delta function in the frequency domain will transform to in the discrete time domain. Let our frequency domain signal be an infinite string of delta functions separated in frequency, Ω, by 2π, or

$$X(\Omega) = \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k). \tag{30}$$

What is the DTFT of Eq. 30? Using the direct method we use Eq. 20 to find

$$\begin{aligned} x(n) &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)\ e^{j\Omega n}\ d\Omega \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{j2\pi k n}. \end{aligned} \tag{31}$$

And, since k and n are always integers, the complex exponential always has an exponent equal to $j2\pi$, meaning that the summation sums an infinite string of ones. Therefore, $x(n) = 1/2\pi$, or

$$1 \rightleftharpoons 2\pi\ \delta(\Omega - 2\pi k). \tag{32}$$

This signifies that an infinite string of delta functions in the n domain transforms to another infinite string of delta functions in the Ω domain.

We now want to determine the DTFT of periodic signals. In the continuous domain we found that $\exp(j\omega_0 t) \rightleftharpoons 2\pi\delta(\omega - \omega_0)$. Similarly, we find that $\exp(j\Omega_0 n) \rightleftharpoons 2\pi\delta(\Omega - \Omega_0 - 2\pi k)$, for all k. The difference is that since Ω is periodic in 2π, we must add the $2\pi k$ term to allow for that periodicity. This relationship can be easily verified by using the techniques used above to find the IDTFT of the delta function,

$X(\Omega)$. Having found this relationship, it is an easy step to proceed to finding the DTFT of a cosine function, A $\cos\Omega_0 n$. We see that if

$$x(n) = A\cos\Omega_0 n = \frac{A}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right), \tag{33}$$

then we also see that

$$X(\Omega) = \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k). \tag{34}$$

Using the same procedure we can find that if

$$x(n) = A\sin\Omega_0 n = \frac{A}{2j}\left(e^{j\Omega_0 n} - e^{-j\Omega_0 n}\right), \tag{35}$$

then

$$X(\Omega) = -j\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k) - \delta(\Omega + \Omega_0 - 2\pi k). \tag{36}$$

We have seen that if we can determine the properties of a transform that we can gain greater insight into, and increase the versatility of the transform. Properties can also make the transform easier to use. Therefore, we will examine a few of the properties of the DTFT.

The major difference between the continuous Fourier transform and the discrete time Fourier transform is that the discrete version is periodic at 2π, or

$$X(\Omega + 2\pi) = X(\Omega). \tag{37}$$

We have seen one ramification of this in that we only had to integrate from 0 to 2π.

Many of the other properties are analogous to the continuous transform properties that we have already seen. Some of these are:

Linearity: $a_1x_1(n) + a_2x_2(n) \rightleftarrows a_1X_1(\Omega) + a_2X_2(\Omega)$

Time shifting: $x(n - n_0) \rightleftarrows \exp(-j\Omega n_0)X(\Omega)$

Frequency shifting: $\exp(j\Omega_0 n)x(n) \rightleftarrows X(\Omega - \Omega_0)$

Scaling: $x(nk) \rightleftarrows X(\Omega/k)$

Differentiation (Ω): $nx(n) \rightleftarrows jX'(\Omega)$

Parseval's theorem exists here as well and we find

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |X(\Omega)|^2 d\Omega. \tag{38}$$

In the continuous domain we saw that convolution was

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\ d\tau, \tag{39}$$

and

$$Y(\omega) = X(\omega)H(\omega). \tag{40}$$

By analogy we can state then that

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)\ h(n-m), \tag{41}$$

$-\infty \le$ **n** $\le \infty$**, and**

$$Y(\Omega) = X(\Omega)H(\Omega). \tag{42}$$

As we already know, many times the input to the system is a sinusoid, whether it be a cosine wave or a sine wave. Whichever it is it can always be described by a linear combination of complex exponentials using the linearity property. Letting

$x(n) = e^{j\Omega n}$, and recognizing that Eq. 41 is equivalent to

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)\ x(n-m), \tag{43}$$

we see that for a complex exponential input we have

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)\ e^{j\Omega(n-m)} = \sum_{m=-\infty}^{\infty} h(m)\ e^{j\Omega n} e^{-j\Omega m}$$

$$= e^{j\Omega n} \sum_{m=-\infty}^{\infty} h(m)\ e^{-j\Omega m} = e^{j\Omega n}\ H(\Omega). \tag{44}$$

This is the steady-state response, $y_{ss}(n)$, of the system, i.e., the response after all the transients have died away. Noting that the exponential multiplier of $H(\Omega)$ is $x(n)$, we can rewrite Eq. 44 as

$$y_{ss}(n) = x(n)\ H(\Omega). \tag{45}$$

This quantity, $H(\Omega)$, which we know is

$$H(\Omega) = \sum_{m=-\infty}^{\infty} h(m)\ e^{-j\Omega m}, \tag{46}$$

is known by several names, such as, the characteristic value, the eigenvalue, or the frequency response of the system.

As an example, let

$$h(n) = \delta(n) + \delta(n-1), \tag{47}$$

and let

$$x(n) = e^{j\frac{\pi}{2}n}, \quad -\infty \le n \le \infty. \tag{48}$$

What is the steady-state response of this system? First, $H(\Omega)$ is

$$H(\Omega) = \sum_{m=-\infty}^{\infty} [\delta(m) + \delta(m-1)]\ e^{-j\Omega m} = 1e^{j0} + 1e^{-j\Omega}$$

$$= 1 + e^{-j\Omega}. \tag{49}$$

From Eq. 48, we see that $\Omega = \pi/2$ and evaluating Eq. 49 at this frequency we get

$$H\left(\frac{\pi}{2}\right) = 1 + e^{-j\frac{\pi}{2}} = 1 - j = 1.41e^{-j\frac{\pi}{4}}. \qquad \textbf{(50)}$$

We can now solve for the steady-state output as

$$y_{ss}(n) = x(n)H(\Omega) = e^{j\frac{\pi}{2}n}\, 1.41e^{-j\frac{\pi}{4}} = 1.41e^{j(\frac{\pi n}{2}-\frac{\pi}{4})}. \qquad \textbf{(51)}$$

Note that the system has changed the amplitude of the input signal from 1 to 1.41, and that the phase has been shifted by $\pi/4$ radians.

C. SAMPLING THEOREM

The purpose of a conversion process has to be to change the signal from analog to discrete form (or vice versa) without loss of information. The link between the two forms, which guarantees recoverability of the information contained in the signal, is the sampling theorem. Sampling is the process of periodically evaluating the value of the signal at the very moment that it is sampled (instantaneous sampling). We will now see how often a signal has to be sampled (i.e., the sampling rate f_s) in order to be guaranteed of recovering the information in that signal.

Let's take an arbitrary analog signal, x(t). If we sample x(t) at a rate of f_s, or every T_s seconds, we would take our first sample at time t = 0, then at time t = T_s, then at time t = $2T_s$, etc. Each time we take a sample we get the instantaneous value of x(t) measured at the moment we take the sample so that we get a series of values x(0), x(T_s), x($2T_s$), etc., which in general would be a series of x(nT_s) values, $-\infty \le$ n $\le \infty$. Multiplying these values by delta functions to hold their places in time will give us the sampled waveform, x(n). In mathematical form x(n) will be the sum of all samples

$$x(n) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s). \qquad \textbf{(52)}$$

This is called the ideal sampled signal.

To determine the requirements of the Sampling theorem, we will look at the implications in both discrete frequency, Ω, and analog frequency, ω. First the discrete case.

Consider a complex exponential input to the ideal sampler, i.e., $x(t) = e^{j\omega t}$. The output of the sampler will then be

$$x(nT_s) = e^{j\omega T_s n} \tag{53}$$

or written as a sequence in discrete frequency

$$x(n) = e^{j\Omega n}, \tag{54}$$

where $\Omega = \omega T_s = 2\pi f T_s$ is the digital or discrete frequency. Assuming that the sampling rate is such that $x(nT_s)$ and $x(n)$ are periodic requires that, for integers k and N

$$\Omega = \omega T_s = \frac{k\ 2\pi}{N}. \tag{55}$$

The analog frequency, ω, can be any value, i.e., $0 \le \omega \le \infty$. With no limits on ω, what are the limits on Ω?

For x(n) to be periodic, we require that

$$x(n+N) = x(n) \tag{56}$$

which, when combining Eq. 56, 55, and 54 we see that

$$\begin{aligned} x(n+N) &= e^{j\Omega(n+N)} = e^{j\Omega n} e^{j\Omega N} \\ &= e^{j\Omega n}\ e^{j(\frac{k2\pi}{N})N} = e^{j(\Omega n + k2\pi)}, \end{aligned} \tag{57}$$

which, simply put says that if x(n) is periodic at N, then Ω is periodic at 2π.

Now, let's use these results to see what is required for two sequences to be distinguishable from each other. Define two sequences $x_1(n) = \exp(j\Omega_1 n)$ and $x_2(n) = \exp(j\Omega_2 n)$. It is clear that these two sequences are identical only if $\Omega_1 = \Omega_2$. But what if $\Omega_2 = \Omega_1 + k2\pi$. According to Eq. 57, these two sequences are

indistinguishable even though are not equal. This is manifested graphically in that the two frequencies will appear at the same locations on the real-imaginary plane. We see that for unambiguous determination of x(n) we must require that

$$-\pi \leq \Omega < \pi. \tag{58}$$

So, what does all this have to do with the Sampling theorem? Recalling that

$$\Omega = \omega T_s = 2\pi f T_s = \frac{2\pi f}{f_s} \tag{59}$$

we can use this relationship and Eq. 58 to see in the analog domain that

$$-\frac{f_s}{2} \leq f < \frac{f_s}{2}. \tag{60}$$

Equation 60 tells us that the frequency of x(t) must be no higher than half the sampling rate (or sampling frequency) in order to have unambiguous recoverability of the signal x(t) from x(n). We can generalize this statement to say that the highest frequency of x(t), f_{max}, cannot exceed $f_s/2$.

Now that we have seen the discrete frequency requirements of the Sampling theorem and the bridge into the analog world, let's now look at the Sampling from the analog point of view.

Let's define our sampled signal as the multiplication of two signals. These two signals are the input signal, x(t), and the instantaneous sampler, δ(t). An infinite string of delta functions separated by the sampling period, T_s, from the sampler, say p(t), is of course

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s). \tag{61}$$

The sampled signal, call it $x_s(t)$, will now be the product of these two signals,

$$x_s(t) = x(t)p(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT_s). \tag{62}$$

From the convolution and duality properties

$$\mathscr{F}[x(t)y(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(u)Y(\omega - u)du, \tag{63}$$

which states that multiplication in the time domain transforms to convolution in the frequency domain. To use this property to find the Fourier transform of Eq. 62, we first find the F.T. of p(t). By the sifting property of the delta function, this will be

$$P(\omega) = \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s). \tag{64}$$

The F.T. of x(t) yields some function in the frequency domain, $X(\omega)$. This function has an amplitude as a function of ω, and since all real signals are time-limited, $X(\omega)$ extends (theoretically) to infinity. However, we define the range that a function extends as the bandwidth, W, of the signal. There are different ways to define the bandwidth, one of which being the point that the amplitude decreases to a value of 0.707 of the maximum value. For this example, we define the bandwidth of the signal to extend from 0 to W. The bandwidth can be thought of as the highest frequency component of the signal x(t).

We can now find $X_s(\omega)$ as

$$\begin{aligned} X_s(\omega) &= \mathscr{F}[x(t)p(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(u)\sum_{n=-\infty}^{\infty} \omega_s \delta(\omega - n\omega_s - u)du \\ &= \frac{\omega_s}{2\pi}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s). \end{aligned} \tag{65}$$

Equation 65 contains more information that you might first think. Let's examine what we can learn from this equation. First of all this is a frequency domain representation, so this the F.T. of our sampled signal (which we know as x(n)). It is, of course, a continuous signal, being defined for all ω. It is repetitive, and repeats at intervals of $n\omega_s$. Each repetition as a mid-frequency value of $\omega_s/2\pi = f_s$. Now, let's let n = 0. This is the frequency domain representation at $X(\omega - 0)$ or $X(\omega)$. Putting in n = 1, we get $X(\omega - \omega_s)$, which is a replica of $X(\omega)$ shifted in frequency out

to $\omega = \omega_s$. Similarly, letting n = -1, replicates $X(\omega)$ at $\omega = -\omega_s$. Since the summation extends over all n, you can see that $X(\omega)$ is replicated an infinite number of times at intervals of ω_s.

We require that the sampling frequency ω_s be greater than 2W, where W is the highest frequency of $X(\omega)$, or

$$\omega_s > 2W. \tag{66}$$

We see that this is equivalent to Eq. 60. This minimum sampling rate of two times the highest frequency is called the Nyquist rate.

Since no signal is strictly band-limited, how can we prevent aliasing? There are two ways. One, filter the signal, $X(\omega)$, to reduce higher frequency components prior to sampling, and two, sample at a higher rate than the Nyquist rate to reduce the proximity of the replicas so that they have less opportunity to overlap.

D. DISCRETE FOURIER TRANSFORM

The Fourier transform has proven to be invaluable in the analysis of linear systems. But, unlike the Fourier series, the transform yields continuous frequency spectra. If we wish to perform analysis of the systems by computer, we have no way of representing these continuous spectra in a form useable by the computer. What we need is a transform which has discrete frequency components, just as in the Fourier series. The answer is the discrete Fourier transform (DFT). Representing the spectra of the transform in discrete form is the key to allowing analysis by computer, and is the essence of digital signal processing.

The derivation of the DFT is simple. The discrete Fourier series gave us every element that we required but was only defined for periodic signals. Let's let a nonperiodic signal become periodic, at period N, but only consider N elements of the newly created periodic signal. We know that periodic time-domain signals result in discrete frequency domain signals.

Looking back to the discrete Fourier series, we multiply both sides of Eq. 10 by N to get a function of frequency, $X(k)$,

$$X(k) = Na_k = \sum_{n=0}^{N-1} x(n)\ e^{-jk\Omega_0 n} = \sum_{n=0}^{N-1} x(n)\ e^{-jk\frac{2\pi}{N}n}. \qquad \textbf{(67)}$$

Equation 9, using X(k), will now be

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\ e^{jk\Omega_0 n} = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\ e^{jk\frac{2\pi}{N}n}. \qquad \textbf{(68)}$$

These equations are now suitable for use on a digital computer. But notice that both n and k vary from 0 to N-1. Therefore, for each value of X(k), N multiplications occur–one for each n. Finding X(k) for all k's will require N^2 multiplications. This can be a large number. For this reason, a more efficient algorithm is normally used, called the Fast Fourier Transform. You have already studied FFTs with Prof. Garcia.

MATLAB uses both DFTs and FFTs. If the number of elements of x is a power of two, the faster FFT is used. Otherwise it uses DFT.